INTEGRALS & ITS APPLICATIONS

1. (A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).
2. (C) $-xe^{-x} + C$
3. (A) $\frac{12^x}{\log 12} + C$
4. (D) log (1 + log x) + C
5. (A) 72
6. (C) $\frac{1}{4}$
7. (B) $\frac{(\log x)^2}{2} + C$
8. (C) $\int_a^b |f(y)|dy$
9. Evaluate

Solution:

(a)

$$\int_0^1 \frac{xe^x}{(x+1)^2}dx = \int_0^1 \frac{(x+1)-1}{(x+1)^2}e^x dx = \int_0^1 \left[\frac{1}{x+1} - \frac{1}{(x+1)^2}\right] dx$$ **[1]**

$$= \left[\frac{1}{x+1}e^x\right]_0^1 = \frac{e-2}{2}$$ $\left[\frac{1}{2} + \frac{1}{2}\right]$

10. If

$$R(x) = \int \left(\frac{6}{(x+2)^2} + 5\right) dx = -\frac{6}{x+2} + 5x + C$$

$$\boxed{5x - \frac{6}{x+2} + C}$$

11. Evaluate:

$$\int_0^4 |x-2|dx = \int_0^2 |x-2|dx + \int_2^4 |x-2|dx$$
$$= \int_0^2 -(x-2)dx + \int_2^4 (x-2)dx$$
$$= \int_0^2 (2-x)dx + \int_2^4 (x-2)dx$$
$$= \left[2x - \frac{x^2}{2}\right]_0^2 + \left[\frac{x^2}{2} - 2x\right]_2^4$$
$$= \left((2(2) - \frac{2^2}{2}) - (2(0) - \frac{0^2}{2})\right) + \left((\frac{4^2}{2} - 2(4)) - (\frac{2^2}{2} - 2(2))\right)$$
$$= \left((4 - \frac{4}{2}) - 0\right) + \left((\frac{16}{2} - 8) - (\frac{4}{2} - 4)\right)$$
$$= (4-2) + ((8-8) - (2-4))$$
$$= (2) + (0 - (-2))$$
$$= 2 + 2$$
$$= 4$$

12. The

Given, $MR = 9 + 2x - 6x^2$

$$TR = \int (9 + 2x - 6x^2)dx$$

$$TR = 9x + x^2 - 2x^3 + k$$

When $x = 0$, $TR = 0$, so $k = 0$

$TR = 9x + x^2 - 2x^3$ 1

$\Rightarrow$ $px = 9x + x^2 - 2x^3$

$\Rightarrow$ $p = 9 + x - 2x^2$ which is the demand function 1

13. The

$$TC = \int (50 + \frac{300}{x+1})dx$$

$TC = 50x + 300\log|x + 1| + k$ 1

If $x = 0, TC = ₹2000$

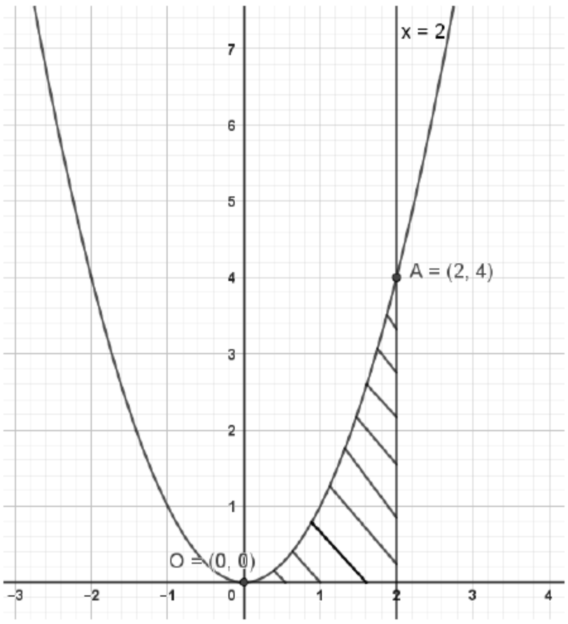
So $2000 = 300(\log 1) + k$ $\Rightarrow k = 2000$

So $TC = 50x + 300\log(x + 1) + 2000$ 1

14. Evaluate:

Sol. Required Area

$= \int_0^2 x^2 dx = \left|\frac{x^3}{3}\right|_0^2 = \frac{8}{3}$ $(1\frac{1}{2})$



$(1\frac{1}{2})$ **for correct graph**

15. Evaluate:

Sol.

$$\mathbf{I} = \int_0^1 \frac{dx}{e^x(1+e^x)}$$

$$= \int_1^e \frac{dt}{t^2(1+t)} \text{ (Put } e^x = t \Rightarrow e^x dx = dt) \qquad \left(\frac{1}{2}\right)$$

$$= \int_1^e \left(-\frac{1}{t} + \frac{1}{t^2} + \frac{1}{1+t}\right) dt \qquad (1)$$

$$= \left[-\log(t) - \frac{1}{t} + \log(1+t)\right]_1^e$$

$$= \left[\log\left(\frac{1+t}{t}\right) - \frac{1}{t}\right]_1^e \qquad (1)$$

$$= \left[\log\left(\frac{1+e}{e}\right) - \frac{1}{e}\right] - [\log 2 - 1]$$

$$= \log\left(\frac{1+e}{2e}\right) - \frac{1}{e} + 1 \qquad \left(\frac{1}{2}\right)$$

16. The

Solution:

(a) Here, $p_0 = 25$

Putting $p = p_0$ and $x = x_0$ in the given supply function $100p = (x + 20)^2$.

We have $x_0 = 30$ **[1]**

Thus,

Producer's surplus $= p_0 x_0 - \int_0^{x_0} p\, dx = 750 - \frac{1}{100}\int_0^{30}(x+20)^2 dx$ **[1]**

$= 750 - \frac{1}{100}\left[\frac{1}{3}(x+20)^3\right]_0^{30}$

$= 750 - \frac{1}{300}[(50)^3 - (20)^3] = ₹\,360$ **[1]**

17. Find :

Solution:

(b) $I = \int \frac{2x^2+1}{x^2-3x+2} dx = \int (2 + \frac{6x-3}{x^2-3x+2}) dx$ **[1]**

$= \int 2dx + \int \left[\frac{-3}{x-1} + \frac{9}{x-2}\right] dx$ {Using partial fractions} **[1]**

$= 2x - 3\log|x-1| + 9\log|x-2| + C$ **[1]**

18. Find :

Sol.

Let $\frac{1}{(x+1)^2(x^2+1)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{Cx+D}{x^2+1}$ ½

Getting the values,

$A = \frac{1}{2}, B = \frac{1}{2}, C = \frac{-1}{2}$ and $D = 0$ 1

Thus

$$I = \int \frac{1}{2(x+1)}dx + \frac{1}{2}\int \frac{1}{(x+1)^2}dx - \frac{1}{2}\int \frac{x\,dx}{x^2+1}$$

$$= \frac{1}{2}\log(x+1) - \frac{1}{2(x+1)} - \frac{1}{4}\log|x^2+1| + C$$ 1½

19. The

20. Evaluate:

$$\int_0^1 \log(1+2x)dx = [\log(1+2x)\cdot x]_0^1 - \int_0^1 x\cdot\frac{1}{1+2x}\cdot 2\,dx$$
$$= [x\log(1+2x)]_0^1 - \int_0^1 \frac{2x}{1+2x}dx$$
$$= (1\cdot\log(1+2(1)) - 0\cdot\log(1+2(0))) - \int_0^1 \frac{(1+2x)-1}{1+2x}dx$$
$$= (\log(3)-0) - \int_0^1 \left(\frac{1+2x}{1+2x} - \frac{1}{1+2x}\right)dx$$
$$= \log(3) - \int_0^1 \left(1 - \frac{1}{1+2x}\right)dx$$
$$= \log(3) - \left[x - \frac{1}{2}\log(1+2x)\right]_0^1$$
$$= \log(3) - \left(\left(1-\frac{1}{2}\log(1+2(1))\right) - \left(0 - \frac{1}{2}\log(1+2(0))\right)\right)$$
$$= \log(3) - \left(\left(1-\frac{1}{2}\log(3)\right) - \left(0 - \frac{1}{2}\log(1)\right)\right)$$
$$= \log(3) - \left(1 - \frac{1}{2}\log(3) - 0\right)$$
$$= \log(3) - 1 + \frac{1}{2}\log(3)$$
$$= \frac{3}{2}\log(3) - 1$$

21. Find:

Q-26 $\int \frac{x^3}{(x+2)}dx = \left(\int x^2 - 2x + 4 - \frac{8}{x+2}\right)dx$ 2 Mark

$= \frac{x^3}{3} - x^2 + 4x - 8\log|x+2| + C.$ 1 *Mark*

(where C is an arbitrary constant of integration)

22. Find:

$\int (x^2+1) ln\, x dx$

Integrating by parts

$lnx\left(\frac{x^3}{3}+x\right)-\int \frac{1}{x}\left(\frac{x^3}{3}+x\right)dx.$ 2 mark

$lnx\left(\frac{x^3}{3}+x\right)-\int \left(\frac{x^2}{3}+1\right)dx$

$lnx\left(\frac{x^3}{3}+x\right)-\left(\frac{x^3}{9}+x\right)$ + C. 1 Mark

23. The

Q-29 Under pure competition

$p_d = p_s$

$\Rightarrow 16 - x^2 = 2x^2 + 4$

$\Rightarrow 3x^2 = 12 \Rightarrow$ x = 2 , -2 ; since x can't be –ve, so x=2 1 Mark

When $x_0 = 2$; $p_0 = 12$ ½ Mark

Hence, Consumer's surplus = $\int_0^2 p_d dx - p_0 x_0$ ½ Mark

24. The

$p_d = p_s$

$\Rightarrow 56 - x^2 = 8 + \frac{x^2}{3}$

$\Rightarrow \frac{4}{3}x^2 = 48 \Rightarrow x^2 = 36 \Rightarrow$ x = 6 , -6 ; since x can't be –ve, so x=6 1 Mark

When $x_0 = 6$; $p_0 = 20$ ½ Mark

Hence, Producer's surplus = $p_0 x_0 - \int_0^6 p_s dx$ ½ Mark

$= 6 \times 20 - \int_0^6 \left(8 + \frac{x^2}{3}\right) dx$

= 120 – [48+24]

= 48 units 1 Mark

25. Evaluate:

$$\int \frac{dx}{(1+e^x)(1+e^{-x})}$$

$$= \int \frac{e^x dx}{(1+e^x)^2}$$ 3

$= \int \frac{dt}{t^2}$, *where* $t = e^x + 1$ *and* $dt = e^x dx$

$$= \frac{-1}{t} + C$$

$$= \frac{-1}{1+e^x} + C$$

26. Evaluate:

$$\int \underset{II}{x}\, \underset{I}{log(1+x^2)}dx$$, Integration by parts

$= \log(1+x^2).\int x dx - \int [\frac{d}{dx}\log(1+x^2).\int x\, dx]\, dx$

$= \frac{x^2}{2}\log(1+x^2) - \int [\frac{2x}{1+x^2}.\frac{x^2}{2}]\, dx$

$= \frac{x^2}{2}\log(1+x^2) - \int \frac{x^3}{1+x^2}\, dx$

$= \frac{x^2}{2}\log(1+x^2) - \int [x - \frac{x}{1+x^2}]\, dx$

$= \frac{x^2}{2}\log(1+x^2) - \frac{x^2}{2} + \frac{1}{2}\log(1+x^2) + C$

$= \frac{1}{2}[(1+x^2)\log(1+x^2) - x^2] + C$

27. Under

Under pure competition, $p_d = p_s$ $\Rightarrow \frac{8}{x+1} - 2 = \frac{x+3}{2}$ $\Rightarrow x^2 + 8x - 9 = 0$ $\Rightarrow x = -9, 1$ $\therefore x = 1$	1.5
When $x_0 = 1 \Rightarrow p_0 = 2$ $\therefore$ Produce surplus = $2 - \int_0^1 \frac{x+3}{2}dx = 2 - [\frac{x^2}{4} + \frac{3x}{2}]_0^1 = \frac{1}{4}$	1.5

28. The

$p = 274 - x^2$ $\Rightarrow R = px = 274x - x^3$ $\frac{dR}{dx} = 274 - 3x^2$ Given MR = $4 + 3x$ In profit monopolist market, MR = $\frac{dR}{dx} \Rightarrow 4 + 3x = 274 - 3x^2$ $\Rightarrow x^2 + x - 90 = 0$	1.5

$\Rightarrow x = -10, 9$ $\therefore x = 9$	
When $x_0 = 9 \Rightarrow p_0 = 193$ $\therefore$ Consumer surplus = $\int_0^9 (274 - x^2)dx - 193 \times 9$ $= [274x - \frac{x^3}{3}]_0^9$ = 486	1.5

29. The

Substituting, $p_0 = ₹48$ in $p = x^2 + 4x + 3$ We get $x_0 = 5$	1
$PS = p_0x_0 - \int_0^{x_0} g(x)dx$ $= 48 \times 5 - \int_0^5 (x^2 + 4x + 3)dx$	1
$= 240 - \left[\frac{x^3}{3} + 2x^2 + 3x\right]_0^5 = ₹133.33$	1

30. If

Sol.

$p = 4 - 5x + x^2$	
$p_0 = 18$, we have $18 = 4 - 5x + x^2$	
or $x^2 - 5x - 14 = 0$	
$\Longrightarrow (x - 7)(x + 2) = 0$	
$\therefore x = 7, x = -2$ is rejected	1½
$p_0x_0 = 18 \times 7 = 126$	½
$PS = p_0x_0 - \int_0^7 (x^2 - 5x + 4)\ dx$	1
$= 126 - \left[\frac{x^3}{3} - \frac{5x^2}{2} + 4x\right]_0^7$	1
$= 126 - \frac{119}{6}$	
$= \frac{637}{6}$ or 106.17 approx.	1

31. Find :

Ans

$$I=\int\frac{x^3}{x^4+3x^2+2}dx$$

Put $x^2=t \Rightarrow x\,dx=\frac{dt}{2}$ ½

$$\therefore I=\frac{1}{2}\int\frac{t}{t^2+3t+2}dt$$ ½

$$=\frac{1}{2}\int\frac{t}{(t+1)(t+2)}dt$$ 1

$$=\frac{1}{2}\int\left(-\frac{1}{t+1}+\frac{2}{t+2}\right)dt$$ 1½

$$=\frac{1}{2}\left[-\log|t+1|+2\log|t+2|\right]+C$$ 1

$$=\frac{1}{2}\left[-\log|x^2+1|+2\log|x^2+2|\right]+C \text{ or } \frac{1}{2}\left[\log\frac{(x^2+2)^2}{x^2+1}\right]+C$$ ½

32. A

Profit = Revenue – Cost $\Rightarrow \frac{dP}{dx}=\frac{dR}{dx}-\frac{dC}{dx}$ **1**

$\Rightarrow \frac{dP}{dx}=$ MR – MC $\Rightarrow \frac{dP}{dx}=-81+36x-3x^2$ **1**

For maximum profit $\frac{dP}{dx}=0 \Rightarrow -81+36x-3x^2=0 \Rightarrow x=3 \text{ or } 9$

Now $\frac{d^2P}{dx^2}=36-6x=36-54<0$, at $x=9$, so P is maximum at $x=9$. **½**

Since $\frac{dP}{dx}=-81+36x-3x^2 \Rightarrow P=\int(-81+36x-3x^2)dx+c$

$\Rightarrow P=-81x+18x^2-x^3+c$ **1**

When $x=0, P=0 \Rightarrow c=0$

$\Rightarrow P=-81x+18x^2-x^3$ and at $x=9, P=-729+2(729)-729=0$ **½**

$\therefore$ Total profit at profit maximizing output is 0. **1**

33. Supply a

Let us consider demand function be $p = D(x) = ax + b........(i)$

When $x = 25$ then $p = 20000$

From equation (i), we have $20000 = 25a + b..........(ii)$ ½

And when $x = 125$ then $p = 15000$

From equation (i), we have $15000 = 125a + b..........(ii)$ ½

On solving equations (i) and (ii), we get $a = -50$ and $b = 21250$ **1**

Therefore, demand function, $p = D(x) = -50x + 21250$ ½

For equilibrium point $D(x_0) = S(x_0)$

$$\Rightarrow -50x_0 + 21250 = 100x_0 + 7000$$

$$\Rightarrow -150x_0 = -14250$$

$$\Rightarrow x_0 = 95$$ ½

On putting value of x_0 in demand function and supply function, we get

$$p_0 = 16500$$ ½

$\therefore$ Consumer surplus (CS)

$$= \int_0^{x_0} D(x)dx - p_0 x_0$$

$$= \int_0^{95} (-50x + 21250)dx - 16500 \times 95$$ **1**

$$= \left(-50\frac{x^2}{2} + 2150x\right)_0^{95} - 1567500$$

$$= 1793125 - 1567500$$

$$= ₹\ 225625$$ ½

34. In the

(i) $p_d = a + bx$

$\Rightarrow 20 = a + 400\,b \ldots$ (i)

and $25 = a + 200\,b \ldots$ (ii) ½

Solving (i) and (ii), we get $a = 30, b = -\frac{1}{40}$

$\therefore\ p_d = 30 - \frac{1}{40}x$ ½

(ii) For equilibrium, $p_d = p_s$

$\Rightarrow 30 - \frac{1}{40}x = -15 + \frac{x}{20}$ ½

$\therefore x = 600$ ½

Equilibrium price $= 30 - \frac{1}{40} \times 600 = ₹\,15$ 1

(iii) (A) Consumer surplus $= \int_0^{600}\left(30 - \frac{1}{40}x\right)dx - 600 \times 15$

$= \left[30\,x - \frac{1}{80}x^2\right]_0^{600} - 9000$

$= 13500 - 90000 = 4500$ 1

$\therefore$ Consumer surplus $= ₹\,4500$

OR

(B) Producer surplus $= 600 \times 15 - \int_0^{600}\left(-15 + \frac{x}{20}\right)dx$ 1

$= 9000 - \left[-15\,x + \frac{1}{40}x^2\right]_0^{600}$

$= 9000 - (-9000 + 9000) = 9000$ 1

$\therefore$ Producer surplus $= ₹\,9000$